2.71

The quarterback $Q$ throws the football when the receiver $R$ is in the position shown. The receiver’s velocity is constant at 10 yd/sec, and he catches passes when the ball is 6 ft above the ground. If the quarterback desires the receiver to catch the ball 2.5 sec after the launch instant shown, determine the initial speed and angle $\theta$ required.

Given: Initial position and speed of receiver. Release point and time
Find: Speed and angle of throw
Assumptions: No acceleration in the x-direction, acceleration in the y-direction is equal to $-g$.
Solution:

\[
x = x_o + v_{ox}t + \frac{a_xt^2}{2}
\]
\[
y = y_o + v_{oy}t + \frac{a_yt^2}{2}
\]
\[x_o = a_x = 0\]
\[y_o = 7 \text{ ft}\]
\[a_y = -g = -32.2 \frac{\text{ft}}{\text{s}^2}\]
\[x = 90 \text{ ft} + 30 \frac{\text{ft}}{\text{s}} (2.5 \text{ s}) = 165 \text{ ft}\]
\[y = 6 \text{ ft}\]
\[\tan \theta = \frac{v_{oy}}{v_{ox}}\]
\[v_o = \sqrt{v_{ox}^2 + v_{oy}^2}\]
So,

\[ x = x_0 + v_{ox}t \]
\[ v_{ox} = \frac{x - x_0}{t} = \frac{165 \text{ ft}}{2.5 \text{ s}} = 66.0 \frac{\text{ft}}{\text{s}} \]
\[ y = y_0 + v_{oy}t - \frac{gt^2}{2} \]
\[ v_{oy} = \frac{y - y_0}{t} + \frac{gt}{2} = -1 \text{ ft} + \frac{32.2 \frac{\text{ft}}{\text{s}} (2.5 \text{ s})}{2} = 39.85 \frac{\text{ft}}{\text{s}} \]
\[ \theta = \arctan \left( \frac{39.85 \frac{\text{ft}}{\text{s}}}{66.0 \frac{\text{ft}}{\text{s}}} \right) = 0.543 \text{ rad (31.1°)} \]
\[ v_o = \sqrt{ \left( 66.0 \frac{\text{ft}}{\text{s}} \right)^2 + \left( 39.85 \frac{\text{ft}}{\text{s}} \right)^2 } = 77.1 \frac{\text{ft}}{\text{s}} \]
2.81
The muzzle velocity of a long-range rifle at A is \( u = 400 \text{ m/s} \). Determine the two angles of elevation \( \theta \) which will permit the projectile to hit the mountain target B.

**Given:** \( y = 1500 \text{ m} \), \( x = 5000 \text{ m} \), \( v_o = 400 \text{ m/s} \).

**Find:** \( \theta \)

**Assumptions:** No acceleration in the x-direction, acceleration in the y-direction is equal to \(-g\).

**Solution:**

\[
x = x_o + v_{ox} t + \frac{a_x t^2}{2} \\
y = y_o + v_{oy} t + \frac{a_y t^2}{2} \\
x_o = y_o = a_x = 0 \\
a_y = -g \\
v_{ox} = v_o \cos (\theta) \\
v_{oy} = v_o \sin (\theta)
\]

Plugging in what we know,

\[
x = v_o \cos (\theta) t \quad (1) \\
y = v_o \sin (\theta) t - \frac{gt^2}{2} \quad (2)
\]

We have two equations and two unknowns. Identifying that we need to solve these two equations is the important part. The rest of the solution is just algebra. There are many ways to handle the algebra. Here are two:
Solving (1) for $t$,

$$t = \frac{x}{v_o \cos \theta} \quad (3)$$

plugging into (2),

$$y = v_o \sin (\theta) \left[ \frac{x}{v_o \cos \theta} \right] - g \left[ \frac{x}{v_o \cos \theta} \right]^2 \quad (4)$$

$$= x \tan \theta - \frac{g x^2}{2 v_o^2} \cos^2 \theta \quad (5)$$

$$= x \tan \theta - \frac{g x^2}{2 v_o^2} \sec^2 \theta \quad (6)$$

$$= x \tan \theta - \frac{g x^2}{2 v_o^2} (1 + \tan^2 \theta) \quad (7)$$

$$0 = -\frac{g x^2 \tan^2 \theta + x \tan \theta - \frac{g x^2}{2 v_o^2} - y}{B \tan^2 \theta + C} \quad (8)$$

$$\tan \theta = -B \pm \sqrt{B^2 - 4AC} \quad 2A \quad (9)$$

where,

$$A = -\frac{g x^2}{2 v_o^2} = -\frac{9.81 \text{ m/s}^2 (5000 \text{ m})^2}{2 (400 \text{ m/s})^2} = -766 \text{ m} \quad (11)$$

$$B = x = 5000 \text{ m} \quad (12)$$

$$C = -\frac{g x^2}{2 v_o^2} - y = -766 \text{ m} - 1500 \text{ m} = -2266 \text{ m} \quad (13)$$

$$\tan \theta_1 = 0.490 \Rightarrow \theta_1 = 0.456 \text{ rad} (21.6\degree) \quad (14)$$

$$\tan \theta_2 = 6.037 \Rightarrow \theta_2 = 1.41 \text{ rad} (80.6\degree) \quad (15)$$

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Here is another way to do the algebra.

Solving (1) for $\cos \theta$,

$$\cos \theta = \frac{x}{v_0 t} \quad (16)$$

Using $\cos^2 \theta + \sin^2 \theta = 1$ to replace $\sin \theta$ in (2),

$$y = v_o t \sqrt{1 - \cos^2 \theta} - \frac{gt^2}{2} \quad (17)$$

plugging (15) into (18)

$$y = v_o t \sqrt{1 - \left( \frac{x}{v_0 t} \right)^2} - \frac{gt^2}{2} \quad (18)$$

$$y = \sqrt{v_o^2 t^2 - x^2} - \frac{gt^2}{2} \quad (19)$$
rearranging
\[ y + \frac{gt^2}{2} = \sqrt{t^2v_o^2 - x^2} \] (20)
squaring both sides
\[ y^2 + ygt^2 + \frac{g^2t^4}{4} = t^2v_o^2 - x^2 \] (21)
rearranging
\[ \frac{g^2}{4}t^4 + (yg - v_o^2)t^2 + x^2 + y^2 = 0 \] (22)
\[ A (t^2)^2 + B (t^2) + C = 0 \] (23)
\[ t^2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \] (24)
with,
\[ A = \frac{g^2}{4} = \frac{(9.81 \text m/s}^2}{4} = 24.1 \text{ m}^2/\text{s}^4 \] (25)
\[ B = yg - v_o^2 = (1500 \text{ m}) \left(9.81 \text{ m/s}^2\right) - \left(400 \text{ m/s}\right)^2 = -145285 \text{ m}^2/\text{s}^2 \] (26)
\[ C = x^2 + y^2 = (5000 \text{ m})^2 + (1500 \text{ m})^2 = 27250000 \text{ m}^2 \] (27)
so,
\[ t^2 = \left(\frac{145285 \text{ m}^2/\text{s}^2}{24.1 \text{ m}^2/\text{s}^2}\right) \pm \sqrt{\left(-145285 \text{ m}^2/\text{s}^2\right)^2 - 4 \left(24.1 \text{ m}^2/\text{s}^2\right) 27250000 \text{ m}^2} \] (28)
\[ t^2 = 193.8 \text{ s}^2, 5845 \text{ s}^2 \] (29)
\[ t_1 = 13.9 \text{ s} \] (30)
\[ t_2 = 76.5 \text{ s} \] (31)
For each time use (1) to find \( \theta \).

\[ \theta = \arccos \left(\frac{x}{v_ot}\right) \] (32)
\[ \theta_1 = \arccos \left[\frac{5000 \text{ m}}{(400 \text{ m/s}) (13.9 \text{ s})}\right] = 0.456 \text{ rad (21.6°)} \] (33)
\[ \theta_2 = \arccos \left[\frac{5000 \text{ m}}{(400 \text{ m/s}) (76.5 \text{ s})}\right] = 1.41 \text{ rad (80.6°)} \] (34)
Height at a range of 5.0 km as a function of launch angle.